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Intermittency in Turbulence: Multiplicative Random Process in Space and Time

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We present a simple stochastic algorithm for generating multiplicative processes with multiscaling both in space and in time. With this algorithm we are able to reproduce a synthetic signal with the same space and time correlation as the one coming from shell models for turbulence and the one coming from a turbulent velocity field in a quasi-Lagrangian reference frame.

KEY WORDS: Turbulence; multifractals; stochastic processes; multiplicative processes.

1. INTRODUCTION

The multifractal language for turbulent flows has been introduced about 20 years ago in order to describe the anomalous scaling properties of turbulence at large Reynolds numbers.[1,2] Beside any particular interpretation, the multifractal formalism exploit the scale invariance of the Navier–Stokes equation by taking into account fluctuations of the scaling exponents. To be more quantitative let us consider the Navier–Stokes equations:

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \nu \, \Delta \vec{u} \tag{1}$$

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where $\vec{u}$ is the velocity field describing a (homogeneous and isotropic) turbulent flow. For $\nu = 0$ the Navier–Stokes equations are invariant with respect to the scale transformation:

$$r \to \lambda r \qquad u \to \lambda^h u \qquad t \to \lambda t^{1-h} \tag{2}$$

Then, following Kolmogorov, it is assumed that at large Reynolds numbers ($\nu \to 0$) the rate of energy dissipation is constant. As a consequence, $h = 1/3$, if no fluctuation on h are present. The above reinterpretation of the Kolmogorov theory naturally opens the way to describe intermittent fluctuations in turbulent flows. Following the original idea by Parisi and Frisch, many possible values of h are allowed in turbulent flows. Each *fluctuation h* at *scale r* is weighted with a probability distribution $P_h(r) \sim r^{3-D(h)}$.

Since its first formulation, the multifractal model of intermittency have been applied to explain many statistical features of intermittency in a unified approach. The final goal of many theoretical investigation is to compute the function $D(h)$ starting by the equation of motions. In some simple although highly nontrivial case such a goal has been recently reached for the case of the Kraichnan model of a passive scalar.[3]

One of the key issue in the multifractal language of turbulence is to understand in a more constructive way what is a multifractal field and how the fluctuations of h are related to the dynamics of the system. In order to develop any systematic theory for computing $D(h)$ starting from the equation of motions, one has to handle a complex nonlinear problem: the way in which a perturbative scheme may be developed strongly depend on a reasonable ansatz on the time-space properties of the probability distribution. It is therefore crucial to understand how we can formulate the most general form of multifractal random field which is consistent with the time and space scaling properties of the Navier–Stokes equations.

One possible interpretation of the multifractal formalism is to observe that for any $r < R$, the multifractal theory predicts:

$$\delta u(r) = W(r, R)\, \delta u(R) \qquad \delta u(r) = u(x+r) - u(x) \tag{3}$$

Then, according to the scaling properties of u, the quantity $W(r, R)$ is a random quantity proportional to $(\frac{r}{R})^h$. It turns out that for $r_1 < r_2 < r_3$ we have

$$W(r_1, r_3) = W(r_1, r_2) \cdot W(r_2, r_3) \tag{4}$$

Equation (4) tells us that one possible interpretation of multifractal field is to assume that fluctuations at scale r are described by a random multiplicative process. The random multiplicative process is also somehow a simple way to mimic the energy cascade in turbulence. Actually, a general formulation of multifractal random fields based on random multiplicative process was first presented in refs. 4 and 5 by using a wavelet decomposition of the field.

One obvious limitation of random multiplicative process is the absence of any time dynamics in the field, as one can immediately highlight by considering space-time correlations. Space-time scaling is a crucial and delicate issue when considering multifractal fields for the Navier–Stokes equations.[8, 9] It is the aim of this paper to understand how one can exhibits a multifractal field whose space and time scaling is consistent with the scaling constraints imposed by the Navier–Stokes equations. In Section 2 we introduce the technical problem shortly reviewed in this introduction by using a rather simplified language. In Section 3 we discuss several implications of the results obtained in Section 2, with a particular emphasis on the consequences for the fusion rules as introduced in refs. 6 and 7. In Section 4 we outline our conclusions and we discuss future extensions of our research.

2. MULTI-SCALE AND MULTI-TIME STOCHASTIC SIGNALS

To simplify even further our argument we can concentrate an a typical fluctuation at a given scale, i.e., disregarding space position. We introduce the scale hierarchy $l_n = l_0 \cdot \lambda^{-n}$, in terms of the scale separation $\lambda > 1$, and the velocity differences $w_n = v(x+l_n) - v(x)$. We assume that the scaling properties of w_n are consistent with the dimensional constraints imposes by the Navier–Stokes equations, i.e.,

$$\partial_t w_n \sim l_n^{-1} w_n^2 \tag{5}$$

If w_n shows multifractal scaling, we may write:

$$\langle w_n^p \rangle \sim \langle w_0 \rangle_0^p \left(\frac{l_n}{l_0} \right)^{\zeta(p)} \tag{6}$$

where $\zeta(p)$ is a nonlinear function of p and $\langle \cdots \rangle_0$ is an average over the large scale statistics. Following Parisi and Frisch, we know that the multifractal scaling (6) can be derived by assuming that $w_n \sim l_n^h$ with probability $l_n^{3-D(h)}$, i.e.,

$$\langle w_n^p \rangle \sim \int dh \, l_n^{ph+3-D(h)} \tag{7}$$

Indeed by means of a saddle point evaluation of the previous integral, one obtain the explicit expression for ζ_p in terms of $D(h)$:

$$\zeta(p) = \inf_h \; [ph + 3 - D(h)] \tag{8}$$

Supposing one wants to keep into account also the time correlations, the constraint (5) implies that

$$C_{p,q}(\tau) = \langle w_n(t)^p \, w_n(0)^q \rangle \sim \int dh \; l_n^{h(p+q)+3-D(h)} f_{p,q}(\tau/\tau_n) \qquad \tau_n = l_n/w_n \tag{9}$$

where τ_n is a random time (the eddy turnover time) and the functions $f_{p,q}$ are dictated by the dynamic equations. Expression (9) has been introduced in ref. 8 and analyzed in details in ref. 9. We underline that, as a consequence of (9), we can predict the scaling properties of quantities like $\frac{d^m C_{p,q}(\tau)}{d\tau^m}$. We now want to understand how to define a random process satisfying both multiscaling in space (6) and multiscaling in time (9). In a more general way, we would like to exhibit random multifractal fields with prescribed dynamical scaling. It is known that the multifractal scaling (6) can be observed for random multiplicative process. Let us introduce the (positive) random variable A_n and let us indicate with $P(A_n)$ the probability distribution of A_n. Then, by defining $w_n = (\prod_{i=1}^n A_i) \, w_0$ and by assuming that the random variables A_i are independent, one obtains:

$$\langle w_n^p \rangle \sim \int \left(\prod_{i=1}^n A_i \right)^p \prod_{i=1}^n P(A_i) \, dA_i = \langle A^p \rangle^n = l_n^{\zeta(p)} \quad \zeta(p) = \log(\langle A^p \rangle)/\log \lambda \tag{10}$$

We want to generalize expression (10) in order to satisfy the dynamical constrain (5). At each scale l_n we introduce the random time $\tau_n = l_n/w_n$.

The generation of our signal proceeds as follows, we extract A_n with probability $P(A_n)$ and keep it constant for a time interval τ_n. Thus, for each scale l_n, we introduce a time dependent random process A_n which is piece-wise constant for a random time intervals τ_n. This is one possible and relatively simple way to take into account the constraint (5).

To give a visual idea of how the algorithm works we show in Fig. 1 the time behaviour of the regeneration times for several scales. It is evident that there are short time-lag where the chain of multipliers is not given by an exact multiplicative process (this happens every time a small scale has to be regenerated but the ancestors are not yet dead).

In Fig. 2 we show the scaling behaviour of the third order structure functions $S_3(l_n) = \langle w_n^3 \rangle$ obtained by a numerical simulation of a time dependent

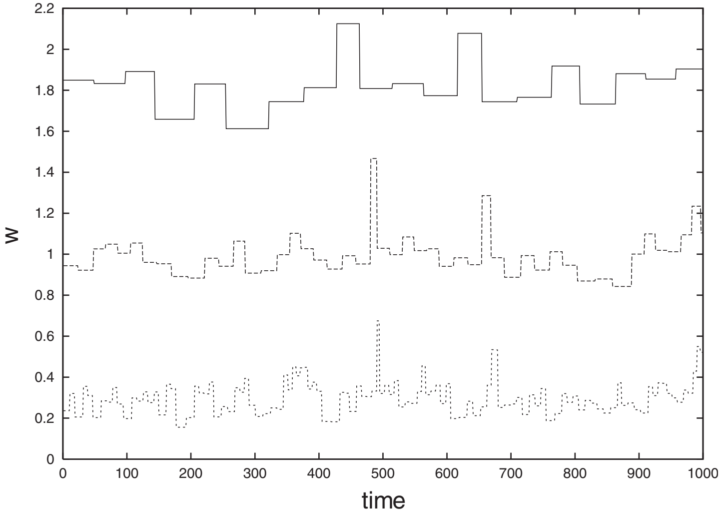

Fig. 1. Time behaviour of w_n for $n = 2, 4, 6$.

random multiplicative process with $P(A) = p_a\delta(A - A_a) + p_b\delta(A - A_b)$, where $A_a = 0.2$, $A_b = 0.6$, $p_b = 1 - p_a$, and p_a has been chosen such that $\zeta(3) = 1$. Although S_3 shows a very well defined scaling, the value of $\zeta(3)$ is greater than what is predicted by (10) (in Fig. 2 the slope -1 is shown for comparison). This effect shows that the "real space" scaling $\langle w_n^p \rangle$ is renormalized by the presence of the nontrivial time dynamics of the multipliers.

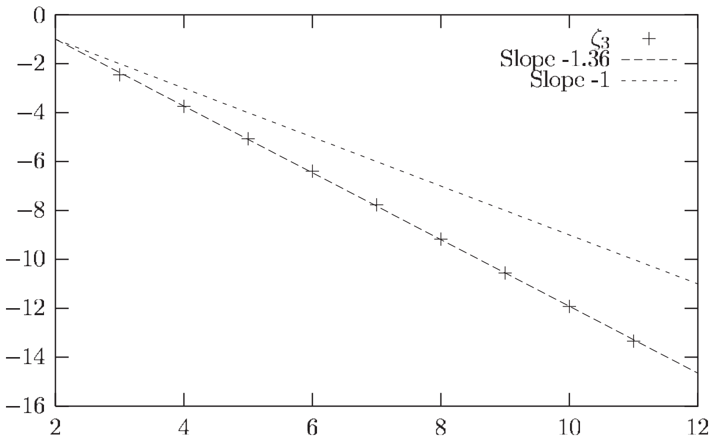

Fig. 2. Log–log plot of measured scaling of $S_3(n)$ vs. n for a time dependent binomial random multiplicative process. The slope fitted slope $\zeta_3 \simeq -1.36$ is "renormalized" with respect to the "bare" value $\zeta_3 = 1$.

In order to understand why (10) cannot be used to predict the scaling exponents, let us understand which is the effect of the time dynamics for a given scale l_n by assuming that at scale l_{n-1} the variable w_{n-1} is kept constant. Let T be the time used for time-averaged quantities and let N_a and N_b be the number of events where the random variable A is equal to A_a and A_b respectively. We next introduce the quantities $\tau_a = l_n/w_n$ and $\tau_b = l_n/w_n$, the times associated to A_a and A_b. By using our definition we can write: $N_a\tau_a + N_b\tau_b = T$, $N_a = p_a N$, $N_b = p_b N$, $N_a + N_b = N$. It then follows:

$$\langle w_n^p \rangle = \langle w_{n-1}^p \rangle \frac{1}{T} \int dt\, A(t)^p = \langle w_{n-1}^p \rangle \frac{1}{T} (\tau_a N_a A_a^p + \tau_b N_b A_b^p) \tag{11}$$

The above expression can be further simplified and we finally obtain:

$$\langle w_n^p \rangle = \langle w_{n-1}^p \rangle \frac{1}{T} \left(\frac{N_a l_n}{A_a w_{n-1}} + \frac{N_b l_n}{A_b w_{n-1}} \right) = \langle w_{n-1}^p \rangle \frac{p_a A_a^{p-1} + p_b A_b^{p-1}}{p_a A_a^{-1} + p_b A_b^{-1}} \tag{12}$$

The consequence of (12) is that the scaling exponents $\zeta(p)$ are renormalized according to the expression:

$$\zeta_R(p) = \zeta_o(p-1) - \zeta_o(-1) \tag{13}$$

where the number $\zeta_o(p)$ are the "bare" scaling exponents, i.e., those computed by using $P(A)$ according to (10).

Expression (13) have been obtained by using the simplified assumption $w_{n-1} = \text{const}$. In the general case, i.e., all variables w_n are fluctuating, one needs to generalize the above discussion. A possible way is to write:

$$\zeta_R(p) = \zeta_o(p-\alpha) - \zeta_o(-\alpha) \tag{14}$$

where the number α (not necessarily integer) depends on the details of $P(A)$.

We have checked expression (14) for a number of different choices of $P(A)$. Here we present the results for $P(A)$ being a log-normal distribution, i.e., for

$$\zeta_o(p) = ph_o - \tfrac{1}{2}\sigma^2 p^2 \tag{15}$$

By using (14) we obtain:

$$\zeta_R(p) = p(h_0 + \alpha\sigma^2) - \tfrac{1}{2}\sigma^2 p^2 \tag{16}$$

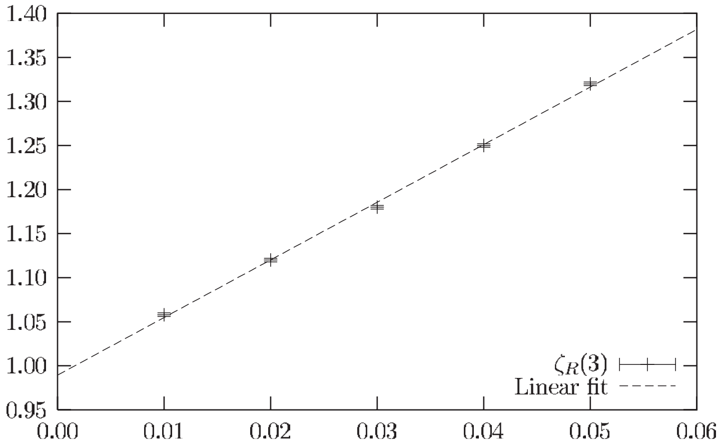

Fig. 3. Behaviour of renormalized $\zeta_R(3)$ for log-normal distribution as a function of σ^2 (see text).

In Fig. 3 we show $\zeta_R(3)$ obtained by a set of numerical simulations for different values of σ and h_0 chosen in such a way that $\zeta_o(3) = 1$. In this case (16) can be written as $\zeta_R(3) = 1 + 3\alpha\sigma^2$. As one can easily observe, the prediction of (16) is verified with very good accuracy with a value of α close to 2. In Fig. 4 we show the value of $\zeta_R(p)$ for $p = 2,..., 6$ as obtained by direct numerical simulation for $\sigma = 0.03$. The dashed line represent the estimate (16) with $\alpha = 2$: a very good agreement is observed.

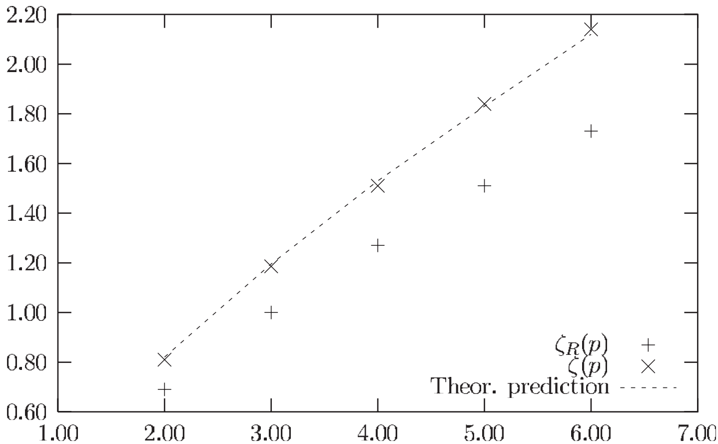

Fig. 4. Behaviour of $\zeta_R(p)$ for $\sigma = 0.03$ (×) as compared to bare exponents $\zeta(p)$. Dashed line is the prediction (16).

The above discussion can be generalized for multifractal fields and for any particular choice of $P(A)$. In all cases, a re-normalization of the scaling exponents, as predicted by (16), should be expected. At the same time, a nontrivial time correlation is introduced for the variables w_n. In Fig. 5 we show $\langle w_n(\tau)\, w_n(0)\rangle$ and $\langle w_{n+6}(\tau)\, w_n(0)\rangle$ for $n=3$, obtained by a numerical simulation of the time dependent random multiplicative process with a log-normal distribution with parameters $\sigma=0.03$, $\zeta_R(3)=1$. As expected, the correlation $\langle w_{n+6}(\tau)\, w_n(0)\rangle$ increases for small τ and then goes to 0. The pick at a time lag larger than zero is due to the presence of a nontrivial time dynamics, i.e., multipliers at different scales need some time to realize that their ancestor have changed their status. This is meant to mimic the non-trivial time dynamics of the turbulent energy transfer. This behaviour is also observed in the numerical simulation of deterministic shell models as reported in ref. 9.

Finally let us check whether the quantities w_n satisfies the scaling constrain imposed by (5). We first observe that the correlation function $\langle w_n(t+\tau)\, w_n(t)\rangle$ goes as $\exp(-B\,|\tau|)$. This is due to the fact w_n as a function of time is not differentiable. In order to check whether (5) is satisfied, we observe that $B \sim k_n w_n^3 \sim$ const. if $\zeta_R(3)=1$. In Fig. 6 we plot the time-scale B as a function of n obtained by a time dependent random multiplicative process, using a log normal distribution with $\sigma=0.03$, $\zeta_o(3)=0.83$, and $\zeta_R(3)=1$. As one can see, B is fairly constant in the inertial range.

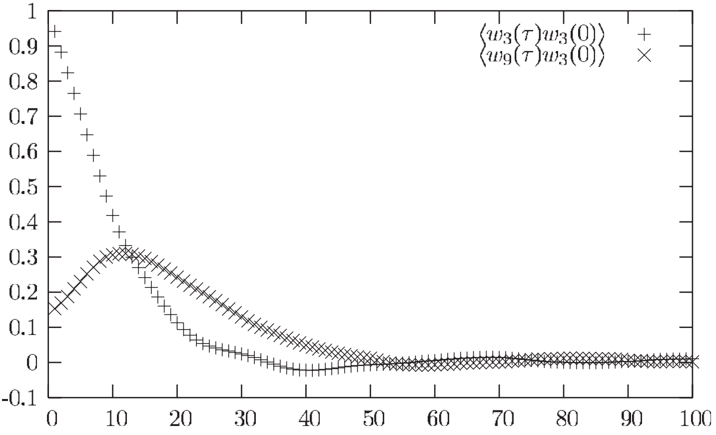

Fig. 5. Time correlations at the same scale, $\langle w_3(\tau)\, w_3(0)\rangle$, and at different scale, $\langle w_9(\tau)\, w_3(0)\rangle$, as a function of the time lag τ.

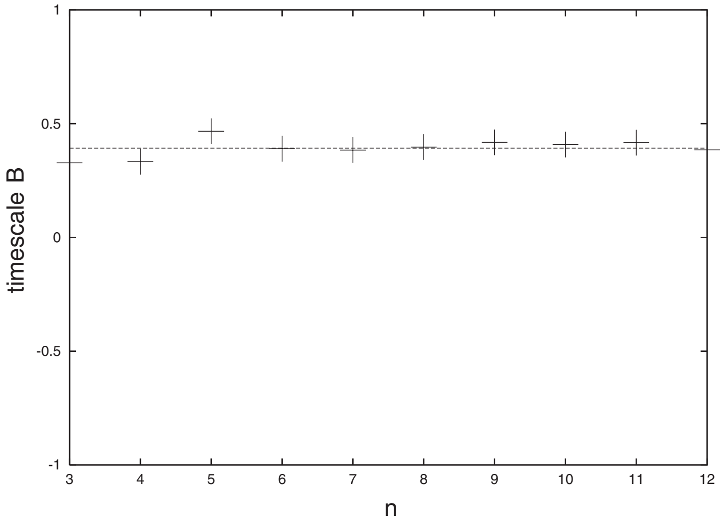

Fig. 6. The time scale B, computed by the expression $\langle w_n(t+\tau)\, w_n(t)\rangle \sim \exp(-B\,|\tau|)$, for different value of n. The scaling constrain (5) should correspond to $B \sim$ const. as observed.

3. NUMERICAL RESULTS

The re-normalization effects discussed in the previous section can be further investigated by considering the case of a passive scalar. In this case the constraint (5) should be written as:

$$\partial\theta_n \sim \theta_n \frac{w_n}{l_n} \tag{17}$$

where $\theta_n = \theta(x+l_n) - \theta(x)$ in analogy with the definition of w_n. We assume that $\langle \theta_n^p \rangle \sim l_n^{H(p)}$. Let us assume that a suitable representation of θ_n is given by a time dependent random multiplicative process as described in Section 2. In this case, however, the random time τ_n is not correlated with the value of θ_n or the related random multiplicative variables. Therefore, we should not expect any re-normalization for the scaling exponents $H(p)$. This is indeed the case as shown in Fig. 7, where we plot the scaling of $\langle \theta_n^3 \rangle$ for a log-normal random multiplicative process with $\sigma = 0.03$ and h_0 such that $H(3) = 1$. The updating times have been chosen with an independent random distribution mimicking the velocity fluctuations, $\tau_n \sim l_n/w_n$ The dashed line corresponds to a slope -1.

Our definition of time dependent random multiplicative process can be very useful in investigating the behaviour of the so called fusion rules

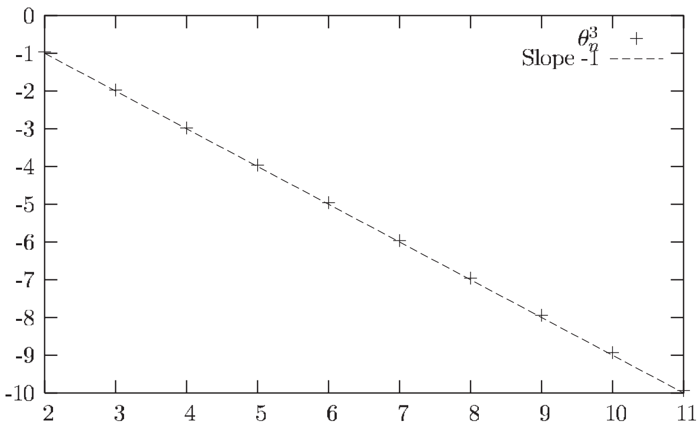

Fig. 7. Log–log plot of $\langle \theta_n^3 \rangle$ vs. n, for the passive scalar as compared to the theoretical prediction with slope -1.

for the quantities like $\langle w_{n+m}^p w_n^q \rangle$. Following refs. 6, 7, and 13, we can write:

$$\langle w_{n+m}^p w_n^q \rangle = C_{p,q}(m) \frac{\langle w_{m+n}^p \rangle}{\langle w_n^p \rangle} \langle w_n^{p+q} \rangle \tag{18}$$

where $C_{p,q}$ is a constant for large m. Actually a direct measurements of $\langle w_{n+m}^p w_n^q \rangle$ in turbulent flows at high Reynolds numbers and in direct numerical simulations, confirm the validity of (18) with $C_{p,q} < 1$ for large m.[13] It is interesting to observe that our time dependent random multiplicative process satisfies fusion rules with $C_{p,q} < 1$. It is indeed possible to show this result by the following argument. Let us consider two scales l_n and l_{n+m}. For fixed time, the quantity w_n and w_{n+m} are not necessary product of the same random variables. They *feel* the same chain of multipliers for some larger scale $l_{n'}$ with $n' < n$. Thus we should have:

$$\langle w_n^q w_{n+m}^p \rangle = \left\langle \prod_{k=1}^{n'} A_k^{q+p} \right\rangle \left\langle \prod_{k=n'}^{n} A_k^q \right\rangle \left\langle \prod_{k=n'}^{n+m} A_k^p \right\rangle \tag{19}$$

where A_k is the random multiplier acting between scale $k-1$ and scale k, i.e., $w_{k+1} = A_k w_k$. The above expression gives the following result for the compensated fusion rules:

$$\frac{\langle w_n^q w_{n+m}^p \rangle \langle w_n^p \rangle}{\langle w_{n+m}^p \rangle \langle w_n^{p+q} \rangle} \sim \frac{\langle A^{q+p} \rangle^{n'} \langle A^q \rangle^{n-n'} \langle A^p \rangle^{m+n-n'} \langle A^p \rangle^n}{\langle A^p \rangle^{n+m} \langle A^{p+q} \rangle^n} \tag{20}$$

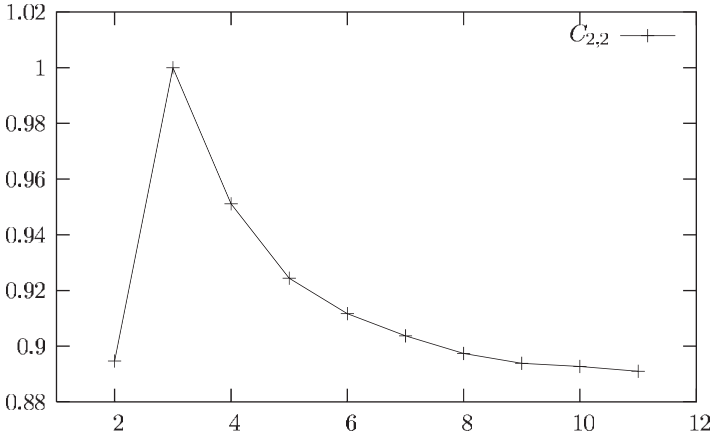

Fig. 8. Behaviour of $C_{2,2}(m)$, defined by using (18) for $n=3$, as a function of $m=-1,..., 11$, for the log-normal probability. For $m=0$, $C_{2,2}(0)=1$, while for large and positive m, $C_{2,2}$ reaches a plateau smaller than 1.

that is:

$$\frac{\langle w_n^q w_{n+m}^p \rangle \langle w_n^p \rangle}{\langle w_{n+m}^p \rangle \langle w_n^{p+q} \rangle} = \left(\frac{\langle A^{q+p} \rangle}{\langle A^q \rangle \langle A^p \rangle} \right)^{n-n'} \leqslant 1 \tag{21}$$

Let us notice that for $n'=n$, which corresponds to random multiplicative process without time dynamics, the r.h.s. of the above expression is just 1. The above equation should be considered as a qualitative prediction. In general we expect n' to be a function of both p and q.

We have checked our prediction by several simulations for different choices of $P(A)$. In Fig. 8 we show the quantity $C_{2,2}(m)$ defined as in (18) for $n=3$ and for $m=-1,..., 11$. As expected $C_{2,2}(m)=1$ for $m=0$ and for $m>0$ is a slowing decaying function of m which reaches a plateau only for very large m.

As a consequence of our analysis, we can also predict that the quantity $C_{p,q}(t)$, defined through:

$$\langle w_{n+m}(t)^p \, w_n(0)^q \rangle = C_{p,q}(t) \, \frac{\langle w_{m+n}^p \rangle}{\langle w_n^p \rangle} \, \langle w_n^{p+q} \rangle \qquad m \gg 1$$

should increase with t. This is indeed the case as one can observe in Fig. 9.

According to (21), the asymptotic value of $C_{p,q}(m)$ for large m is a function of the intermittency, i.e., $C_{p,q}(\infty)$ becomes smaller for larger value of the anomaly $\zeta_R(p)+\zeta_R(q)-\zeta_R(p+q)$. The qualitative prediction of (21)

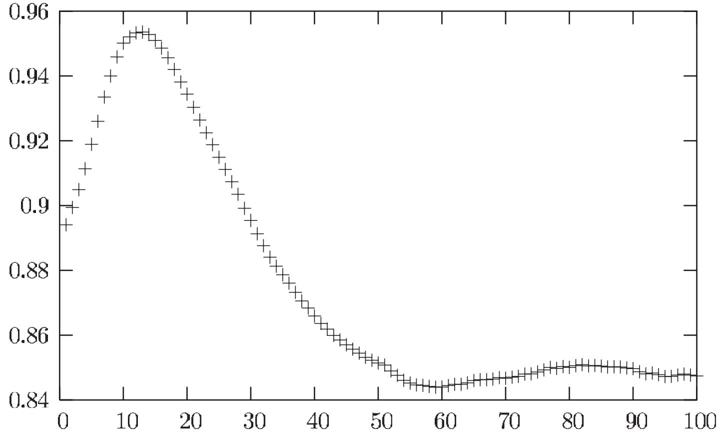

Fig. 9. Behaviour of $C_{9,3}(t)$ as a function of time.

has been checked against numerical simulations of time dependent random multiplicative process. In Fig. 10 we plot the asymptotic value of $C_{2,2}(\infty) = \lim_{m\to\infty} C_{2,2}(m)$ for a log normal $P(A)$ as a function of $\zeta_R(4)-2\zeta_R(2)$. As one can see the qualitative prediction is confirmed.

A further inspection of the numerical simulations reveals that $C_{p,q}$ can be written as

$$C_{p,q}(m) = 1 - \Delta_{p,q} f(m) \tag{22}$$

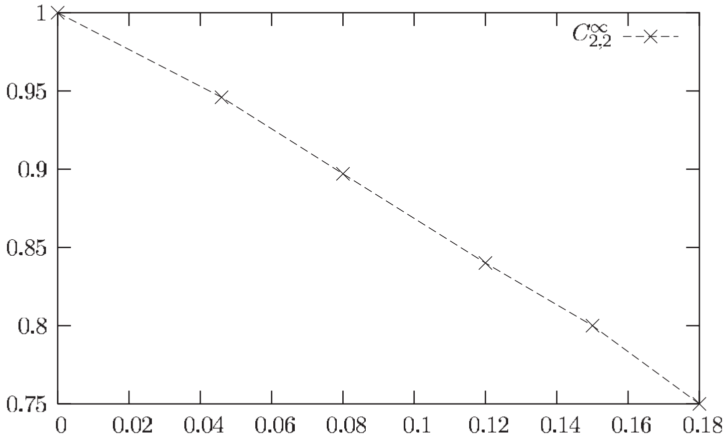

Fig. 10. Behaviour of $C_{2,2}(\infty)$ vs. $\zeta_R(4)-2\zeta_R(2)$.

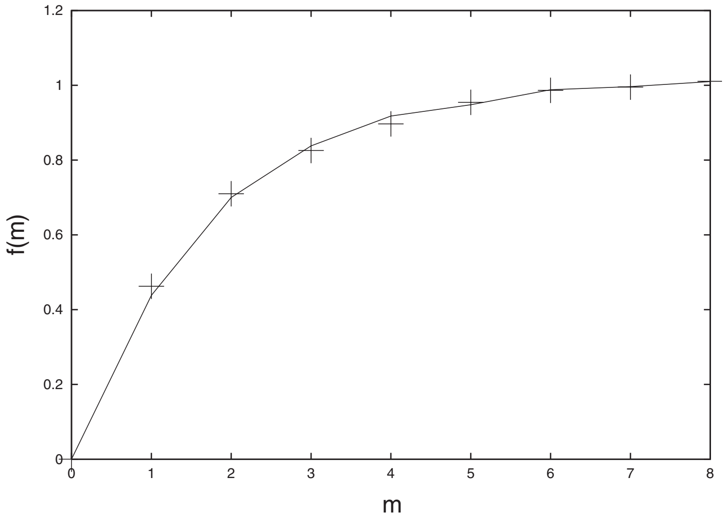

Fig. 11. Behaviour of $f(m)$ as a function of m for a log-normal distribution and two different values of σ, namely $\sigma = 0.03$ (crosses) and $\sigma = 0.05$ (full line).

where $f(0) = 0$ and $f(\infty) = 1$. While $\Delta_{p,q}$ is a function of $\zeta_R(p) + \zeta_R(q) - \zeta_R(p+q)$, we may wonder whether the function $f(m)$ is somehow universal. Although a definite conclusion cannot be reached by looking at the numerical simulations, still our results seem to indicate that $f(m)$ is either universal or is a function weakly dependent on intermittency. This can be seen from Fig. 11 where we plot $f(m)$ for different values of $\zeta_R(4) - 2\zeta_R(2)$.

We already observed at the beginning of this section that a time dependent multiplicative process for a passive scalar does not show a re-normalization of the scaling exponents $H(p)$. However, the argument used to derive (21) can be applied even if the random times τ_n are not correlated to the multiplicative process, as in the case of the passive scalar. In Fig. 12, we show the quantity $G(m)$ defined by the relation:

$$\langle \theta_3^2 \theta_{3+m}^2 \rangle = G(m) \frac{\langle \theta_{3+m}^2 \rangle}{\langle \theta_3^2 \rangle} \langle \theta_3^2 \rangle \tag{23}$$

for the passive scalar and $m = -1, \ldots, 11$. As expected, the fusion rules are satisfied only asymptotically with prefactors $G(m) < 1$.

Let us also notice that the coefficients $C_{p,q}$ of the fusion rules are not fully defined, in terms of the time dependent random multiplicative process. Let us consider the variable

$$u_n = g_n w_n$$

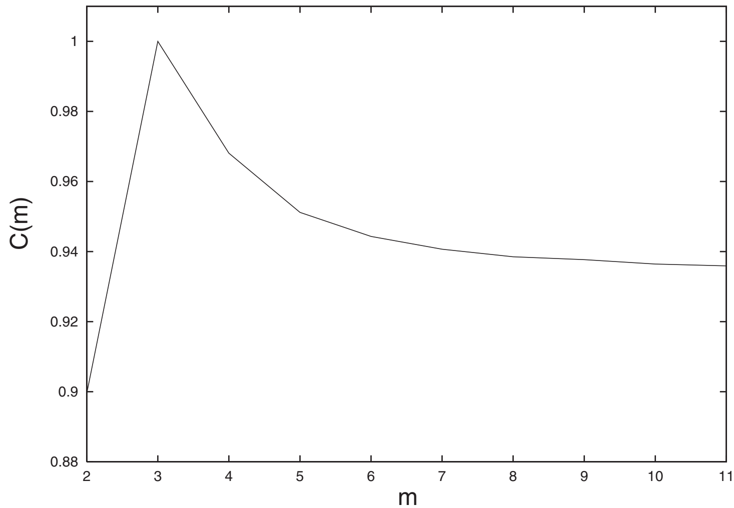

Fig. 12. Plot of the fusion rule coefficient $G(m)$, as defined in the text, for the passive scalar and for $m = -1, \ldots, 11$.

where g_n is a random variable, independent of n, with the same probability distribution for any n, i.e., $\langle g_{n_1} g_{n_2} \cdots g_{n_k} \rangle = \langle g_{n_1} \rangle \langle g_{n_2} \rangle \cdots \langle g_{n_k} \rangle$ and $P(g_n)$ does not depend on n. It follows that the fusion rules for u_n satisfy:

$$\langle u^p_{n+m} u^q_n \rangle = \langle g^p \rangle \langle g^q \rangle \langle w^p_{n+m} w^q_n \rangle$$

which gives

$$\langle u^p_{n+m} u^q_n \rangle = \langle g^p \rangle \langle g^q \rangle \, C_{p,q} \frac{\langle u^p_{m+n} \rangle}{\langle u^p_n \rangle} \langle u^{p+q}_n \rangle \frac{1}{\langle g^{p+q} \rangle}$$

Note that, for each n, w_n is the time dependent multiplicative process and that the scaling properties of u_n and w_n are the same. The above equation implies that the fusion rules coefficients $C_{p,q}$ for u_n becomes:

$$C_{p,q} \to C_{p,q} \frac{\langle g^p \rangle \langle g^q \rangle}{\langle g^{p+q} \rangle} \qquad w_n \to g_n w_n \tag{24}$$

Equation (24) implies that the asymptotic value of $C_{p,q}$ for large m is fixed up to a number (less than 1) linked to a scale invariant probability distribution function.

Before closing this section we would like to notice that the renormalized exponents $\zeta_R(p)$ are associated to a less intermittent field with respect to the bare exponents $\zeta_o(p)$. This can easily be deduced by using Eq. (14). Thus we can predict that whenever the random times τ_n are not correlated to w_n we should observe (for a given $P(A)$) an increase of intermittency. One may wonder whether this qualitative prediction may have any experimental evidence. This is indeed the case. Let us consider a shear flow. It has been noted that whenever the mean shear $\mathscr{S}$ is large enough intermittency increases. For large scale the characteristic time scale of the dynamics should be dominated by the shear effect and we would expect the characteristic time scale independent on the scale.[10–12] In this case, therefore, if we describe intermittency as a random multiplicative process, time dynamics does not lead to a re-normalization of the scaling exponents, i.e., we should observe the scaling exponents $\zeta_o(p) < \zeta_R(p)$. It is suggestive to think that the increase of intermittency in shear dominated flows may be understood in terms of the absence of re-normalization in time dependent multiplicative process.

4. CONCLUSIONS AND DISCUSSIONS

We have introduced a simple multiplicative process which embeds intermittency both in time and in space. This allow us to generate a signal which respect the constraint (imposed by the Navier–Stokes equations) given by Eq. (5): $\partial_t w_n \sim l_n^{-1} w_n^2$. This models is a generalization of the multiplicative process (10) and a practical implementation of a signal satisfying the multifractal representation (9). Studying the numerical process we found a "re-normalization" of the scaling exponents in space due the non-trivial interplay between multipliers re-generation and time evolution. We have shown that in the case of a passive scalar this effect is not present. Furthermore, we have clearly connected the asymptotic gap, observed on fusion rules, with the intermittent scaling exponents. This finding will be exploited in a forthcoming paper[14] to build up a stochastic closure to compute anomalous scaling exponents for shell models of turbulence.

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